15.7 Relationship among Delta, Theta, and Gamma

$\Theta = \frac{\partial \Pi}{\partial t}; \quad \Delta = \frac{\partial \Pi}{\partial S}; \quad \Gamma = \frac{\partial^2 \Pi}{\partial S^2}.$

- It follows that: $\Theta + rS\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma = r\Pi$
- For a delta-neutral portfolio, Δ = 0 and $\Theta + \frac{1}{2}\sigma^2 S^2 \Gamma = r\Pi$

This shows that when Θ is large and positive, gamma tends to be large and negative, and vice versa. This is consistent with the way in which Figure 15.8 has been drawn and explains why theta can be regarded as a proxy for gamma.

15.8 Vega

The *vega* of a portfolio of derivatives, V, is the rate of change of the value of the portfolio with respect to the volatility of the underlying asset:

$$V = \frac{\partial \Pi}{\partial \sigma}$$

If vega is high in absolute terms, the portfolio's value is very sensitive to small changes in volatility (and vice versa).

A position in the underlying asset or in a forward contract has zero vega.

The vega of a portfolio changes by adding a position in a traded option.

If V is the vega of the portfolio and V_T is the vega of a traded option, a position of $-V/V_T$ in the traded option makes the portfolio instantaneously vega neutral.

A portfolio that is gamma neutral will not, in general, be vega neutral, and vice versa .

For a portfolio to be both gamma and vega neutral, at least two traded derivatives dependent on the underlying asset must usually be used.

Example 6: You are given a portfolio that is delta neutral. In addition, you are given the following:

Symbol	Description
$\Gamma = -5{,}000$	The gamma of the portfolio is -5,000
$V = -8{,}000$	The vega of the portfolio is -8,000
$\Gamma_{t1} = 0.5$	The gamma of traded option 1 is 0.5
$\Gamma_{t2} = 0.8$	The gamma of traded option 2 is 0.8
$V_{t1} = 2.0$	The vega of traded option 1 is 2.0
$V_{t2} = 1.2$	The vega of traded option 2 is 1.2
$\Delta_{t1} = 0.6$	The delta of traded option 1 is 0.6
$\Delta_{t2} = 0.5$	The delta of traded option 2 is 0.5
w_i	The amount of traded option i.

1. To make the portfolio vega neutral, take a long position of $w_1 = \frac{V}{V_{t1}} = \frac{8,000}{2} = 4,000$ in traded options.
2. To maintain delta neutrality, 2,400 units of the asset must be sold, since delta increases to 2,400.
   Note: The gamma of the portfolio changes from -5,000 to -3,000.
3. To make the portfolio gamma and vega neutral, we need to determine the w_1 and w_2 to be included in the portfolio:
   $-5,000 + 0.5w_1 + 0.8w_2 = 0$
   $-8,000 + 2.0w_1 + 1.2w_2 = 0$
   The solution is $w_1 = 400$, $w_2 = 6,000$.
4. The delta of the portfolio after adding the positions in the two traded options is 400 * 0.6 + 6,000 * 0.5 = 3,240. Hence 3,240 units of the asset would have to be sold to maintain delta neutrality.

Formulas for Vega:

1. A European call or put option on a non-dividend-paying stock: $V = S_0\sqrt{T}N'(d_1)$, where

$$d_1 = \frac{\ln(\frac{S_0}{K}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T} \text{ and } N'(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$$

2. A European call or put option on a stock or stock index paying a continuous dividend yield at rate q,

$$V = S_0\sqrt{T}N'(d_1)e^{-qT}, \text{ where } d_1 = \frac{\ln(\frac{S_0}{K}) + (r - q + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

The above equation gives the vega for:

a. a European currency option with q equal to r_f
b. a European futures option with q equal to r and S_0 equal to F_0

Notes:

- The vega of a long position in a call or put option is always positive.
- Gamma neutrality protects against large changes in the price of the underlying asset between hedge rebalancing.
- Vega neutrality protects against variations in σ.

15.9 RHO

The *rho* of a portfolio of derivatives measures the sensitivity of the portfolio value to interest rates.

$$rho = \partial\Pi / \partial r$$

A. For a European call option on a non-dividend-paying stock, $rho = KTe^{-rT}N(d_2)$

B. For a European put option on the stock, $rho = -KTe^{-rT}N(-d_2)$, where

$$d_2 = d_1 - \sigma\sqrt{T} \text{ and } d_1 = \frac{\ln(\frac{S_0}{K}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}.$$

These same formulas apply to European call and put options on stocks and stock indices paying a dividend yield at rate *q*.

In the case of currency options, there are two rhos corresponding to the two interest rates.

1. The rho corresponding to the <u>domestic</u> interest rate is given by previous formulas.
2. The rho corresponding to the <u>foreign</u> interest rate for a European call on a currency is given by $rho = -Te^{-r_f T} S_0 N(d_1)$, and, for a European put it is $rho = Te^{-r_f T} S_0 N(-d_1)$

Section 15.5 – 15.9 Review

Given the following: $S_0 = 305;\ \mathrm{K} = 300;\ q = .03;\ \mathrm{r} = .08;\ \sigma = .25;\ T = \frac{1}{3}$, compute the theta, gamma, vega, and rho of a 4 Month European Put Option on a Stock Index:

Preliminary Calculations: compute d_1 , d_2 , $N'(d_1)$, $N(-d_1)$ and $N(-d_2)$

$$d_1 = \frac{\ln(\frac{S_0}{K}) + (r - q + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}.\quad d_1 = \frac{\ln(\frac{305}{300} + (.08 - .03 + \frac{(.25)^2}{2}).\bar{3}}{.25\sqrt{.\bar{3}}} = .30215;\ d_2 = d_1 - \sigma\sqrt{T} = .1578$$

$$N'(.30215) = \frac{1}{\sqrt{2*3.1415}} e^{\frac{-(.30215)^2}{2}} = .3811.\quad N(-.30215) = .3974 \text{ and } N(-.1578) = .4370$$

1. Theta. $\Theta = \frac{-S_0 N'(d_1)\sigma e^{-qt}}{2\sqrt{T}} - qS_0 N(-d_1)e^{-qT} + rKe^{-rT}N(-d_2);\ N'(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$

$$= \frac{-305 * \frac{1}{\sqrt{2*3.1415}} e^{\frac{-(.30215)^2}{2}} * .25 * e^{-.03(.\bar{3})}}{2\sqrt{.\bar{3}}} - .03 * 305 * .3794 * e^{-.03(.\bar{3})} + .08(300)e^{-.08(\bar{3})} * .4370$$

$$= -24.915 - 3.4369 + 10.212 = -18.14$$

When time is measured in days, theta is -18.14/252 = - 0.072.

2. Gamma. $\Gamma = \frac{N'(d_1)e^{-qT}}{S_0\sigma\sqrt{T}} = \frac{\frac{1}{\sqrt{2*3.14159}} e^{-\frac{(.30215)^2}{2}} * e^{-.03(.\bar{3})}}{305 * .25 * \sqrt{.\bar{3}}} = \frac{.3811 * .99}{44.022} = .0085705$

An increase of 1.0 (from 305 to 306) in the index increases the delta of the option by 0.00857.

3. Vega. $V = S_0\sqrt{T}N'(d_1)e^{-qT} = 305\sqrt{.\bar{3}} * (\frac{1}{\sqrt{2*3.1415}} e^{-\frac{(.30215)^2}{2}}) * e^{-.03(.\bar{3})}$

$$= 176.0918 * .381 * .99 = 66.44$$

For a 1% increase in σ (from 25% to 26%), the option increase by 0.6644 (= 0.01 * 66.44).

4. $\boldsymbol{RHO} = -KTe^{-\Gamma T} * N(-d_2) = -300(.\bar{3})e^{-.08(.\bar{3})} * N(-.69784)$

$$= -97.368 * .4370 = -42.54$$

For a 1% increase in $r_f (.08 \to .09)$, the option decreases by $(.01)(42.54) = .4254$

For more information, see examples 15.3, 15.5, 15.7, and 15.8.